EIGENVALUES & EIGENVECTORS

Eigenvalues

Let $\mathbf{A}$ be a $p \times p$ square matrix and $\mathbf{I}$ be a $p \times p$ identity matrix.
Eigenvalues are the scalars $\lambda_1, \lambda_2, \ldots, \lambda_p$ that satisfy the polynomial $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}, \text{ so that } |\mathbf{A} - \lambda \mathbf{I}| = \left| \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0$$

This result implies that there are two solutions to this polynomial or two roots:
$\lambda_1 = 1$ and $\lambda_2 = 3$. These are the Eigenvalues of matrix $\mathbf{A}$.

Another resource can be found at: http://mathworld.wolfram.com/Eigenvalue.html

Eigenvectors

Let $\mathbf{A}$ be a $p \times p$ square matrix and λ be an eigenvalue of $\mathbf{A}$. If $\underline{x}$ is a nonzero vector, where

$$\mathbf{A}\,\underline{x} = \lambda\,\underline{x}$$

Notice that we can rearrange terms by $\mathbf{A}\,\underline{x} - \lambda\,\underline{x} = 0$, so that $[\mathbf{A} - \lambda\mathbf{I}]\,\underline{x} = 0$.
In order to obtain a nonzero solution, the determinant $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

$\underline{x}$ is an eigenvector of matrix $\mathbf{A}$ associated with eigenvalue λ.

Since $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$, with $\lambda_1 = 1$ and $\lambda_2 = 3$, the associated eigenvectors can be determined:

$\mathbf{A}\,\underline{x} = \lambda_1\,\underline{x}$ $\quad \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, then we have:

$\begin{bmatrix} 1 & 0 \\ & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ \ \end{bmatrix} \rightarrow$ $\quad 1(x_1) + 0(x_2) = 1(x_1)$

which resolves to $x_1 = x_1$ and

$\begin{bmatrix} & \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1 \begin{bmatrix} \ \\ x_2 \end{bmatrix} \rightarrow$ $\quad 1(x_1) + 3(x_2) = 1(x_2)$

Which resolves to $x_1 + 3x_2 = x_2$

$\mathbf{A}\,\underline{x} = \lambda_2\,\underline{x}$ $\quad \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, which resolves to $x_1 = 3x_1$ and $x_1 + 3x_2 = 3x_2$

You can see that the number of solutions would be large.
If we arbitrarily set $x_2 = 1$, where $x_1 + 3x_2 = x_2$ then we can solve for $x_1 + 3(1) = 1$ so $x_1 = -2$.

This yields $\underline{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ for the first system and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for the second.

Usually, eigenvectors are determined in such a way to obtain a length of one, where $\underline{x}$ is normalized so that $\underline{x}'\underline{x} = 1$. Normalized eigenvectors are typically denoted by $\underline{e}$.

If $\mathbf{A}$ is a square matrix of order p, then $\mathbf{A}$ has p pairs of eigenvalues and eigenvectors, λ_1, $\underline{e}_1$; λ_2, $\underline{e}_2$; …, λ_p $\underline{e}_p$. The eigenvectors are typically chosen so that $\underline{e}'\underline{e} = 1$ and are orthogonal.

Positive Definite Matrices

Multivariate statistics, for the most part, involves the study of variation and interrelationships. These statistics are founded on the assumption that data are multivariate normally distributed with a corresponding distance in the multivariate space. Distances (multivariate variance) and the multivariate normal density are frequently expressed in terms of matrix products called quadratic forms.

When investigating associations among vectors, we use vector decomposition and can do this in a two dimensional Euclidian orthogonal space. When we involve matrices, we are talking about multiple dimensions that are difficult (if not impossible) to represent geometrically. With matrices, we call this method spectral decomposition.

Spectral decomposition of a symmetric matrix, where $\mathbf{A} = \mathbf{A}'$, is

$$\mathbf{A} = \lambda_1 \underline{e}_1 \underline{e}'_1 + \lambda_2 \underline{e}_2 \underline{e}'_2 + \ldots + \lambda_p \underline{e}_p \underline{e}'_p$$

Where lambdas are eigenvalues and $\underline{e}$ is the associated normalized eigenvector.

Consider $\underline{x}' \mathbf{A} \underline{x}$ as a quadratic form because it only has squared terms x_i^2 and product terms $x_i x_j$.

If we have a symmetric matrix $\mathbf{A}$ such that $0 \le \underline{x}' \mathbf{A} \underline{x}$ for all $\underline{x}' = (x_1, x_2, \ldots, x_p)$, the matrix $\mathbf{A}$ is *nonnegative definite*.

If the equality $0 \le \underline{x}' \mathbf{A} \underline{x}$ is true only for the vector $\underline{x}' = (0, 0, \ldots, 0)$, then $\mathbf{A}$ is said to be *positive definite*.

Matrix $\mathbf{A}$ is positive definite if $0 < \underline{x}' \mathbf{A} \underline{x}$ for all vectors $\underline{x} \neq 0$.

Example

Consider the following quadratic form:

$$3x_1^2 + 2x_2^2 - 2\sqrt{2}x_1x_2 \text{ in matrix notation: } \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{x}' \mathbf{A} \, \underline{x}$$

We can find the eigenvalues:

$$| \mathbf{A} - \lambda \mathbf{I} | = \left| \begin{bmatrix} 3 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 3-\lambda & -\sqrt{2} \\ -\sqrt{2} & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2 = 0$$

The solutions include $\lambda_1 = 4$ and $\lambda_2 = 1$. Here is the work:

$$\begin{array}{ll} (3\text{-}4)(2\text{-}4) & -2 = 0 \\ (\text{-}1)(\text{-}2) & -2 = 0 \\ (2) & -2 = 0 \end{array} \qquad \text{and} \qquad \begin{array}{ll} (3\text{-}1)(2\text{-}1) & -2 = 0 \\ (2)(1) & -2 = 0 \\ (2) & -2 = 0 \end{array}$$

These, $\lambda_1 = 4$ and $\lambda_2 = 1$, are the Eigenvalues of matrix $\mathbf{A}$.

Employing spectral decomposition, we find

$$\mathbf{A} = \lambda_1 \underline{e}_1 \underline{e}'_1 + \lambda_2 \underline{e}_2 \underline{e}'_2$$
$$\mathbf{A} = 4 \, \underline{e}_1 \underline{e}'_1 + \underline{e}_2 \underline{e}'_2$$

Where $\underline{e}_1$ and $\underline{e}_2$ are the normalized orthogonal eigenvectors associated with the eigenvalues.

We then have the result

$$\underline{x}' \mathbf{A} \, \underline{x} = 4 \, \underline{x}' \underline{e}_1 \underline{e}'_1 \underline{x} + \underline{x}' \underline{e}_2 \underline{e}'_2 \underline{x} \qquad \text{where dimensions are (1x2)(2x1)(1x2)(2x1)}$$

$$= 4 \, y_1^2 + y_2^2 \geq 0 \qquad \text{where } y_1 = \underline{x}' \underline{e}_1 = \underline{e}'_1 \underline{x} \text{ and } y_2 = \underline{x}' \underline{e}_2 = \underline{e}'_2 \underline{x}$$

With this notation, we see that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \underline{e}'_1 \\ \underline{e}'_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{y} = \mathbf{E} \, \underline{x}$$

$\mathbf{E}$ is an orthogonal matrix and so it has an inverse. So, $\underline{x} = \mathbf{E}^{-1}\underline{y}$.

$\underline{x}$ is a nonzero vector, so $0 \neq \underline{x} = \mathbf{E}^{-1}\underline{y}$ suggests that $\underline{y} \neq 0$.

This is not a procedure that is done by hand. We will use the matrix algebra to complete spectral decomposition, to estimate eigenvalues and eigenvectors.